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COMMENT

Bond percolation critical probability bounds derived by edge contraction

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Abstract. A contraction principle is valid for bond percolation models. If G^c is obtained from G by contraction of a set of edges, then the bond percolation critical probabilities satisfy $p_c(G^c) \leq p_c(G)$. The contraction principle provides relationships between graphs which do not follow from the inclusion principle. Application of the principle provides the following bounds:

$$0.5000 \leq p_c(\text{pentagon lattice}) \leq 0.6527$$

$$p_c(\text{Kagomé lattice}) \leq 0.6180$$

$$0.3820 \leq p_c(\text{dice lattice}).$$

For many years following the introduction of percolation models by Broadbent (1954) and Broadbent and Hammersley (1957), the major focus of mathematical percolation theory was the exact determination of critical probabilities (also called percolation thresholds). A heuristic method of Sykes and Essam (1964) conjectured the value of $\frac{1}{2}$ for the square lattice bond model and the triangular lattice site model, $2\sin(\pi/18)$ for the triangular lattice bond model, and $1-2\sin(\pi/18)$ for the hexagonal lattice bond model. Rigorous proofs for these values were obtained much later by Kesten (1980, 1982) and Wierman (1981). Kesten's (1982) principal theorem verified that the bond percolation critical probabilities of dual lattices sum to one (for periodic lattices with at least one axis of symmetry). Wierman (1984) derived exact bond percolation critical probabilities for an additional pair of dual lattices, using the star-triangle transformation. However, exact critical probabilities are known for only a small set of two-dimensional periodic graphs.

There are also few techniques for deriving rigorous bounds for the critical probability in percolation models. The reciprocal of the connectivity constant (determined from the numbers of self-avoiding paths in the lattice) is a lower bound for the critical probability of both the bond and site models (see Hammersley 1957). The inclusion principle (see Fisher 1961) can be used to find upper and lower bounds by comparison with lattices for which the critical probabilities are known.

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In this comment, we introduce a technique for determining bounds for bond percolation models, based on contraction of edges. The operation of contraction of an edge involves removing it from the graph and replacing the endpoints by a single vertex, with all the other edges incident on the two endpoints being retained. In the bond percolation model, this is equivalent to raising the probability that each contracted edge is open from p to 1. From this fact, it is clear that, if G^c is obtained by contraction of edges in G, then $p_c(G^c) \leq p_c(G)$. The fact also shows that the contraction principle is analogous to the inclusion principle. Obtaining G^d from G by deletion of edges (so $G^d \subset G$) is equivalent to lowering the edge probabilities from p to 0 on the deleted edges.

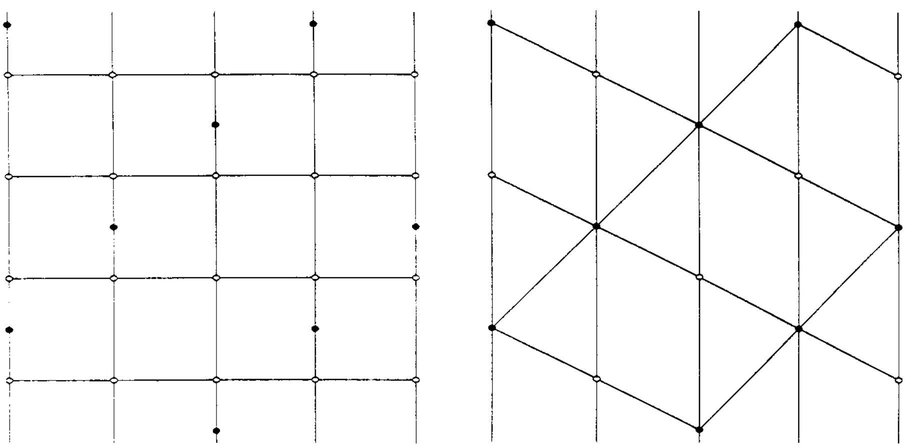
Contraction of edges is a common tool in graph theory, which has been applied in percolation theory before. The fact that the bond percolation critical probability of a lattice is no greater than the site percolation critical probability of the lattice may be proved using the contraction–deletion method (see Kesten 1982, §10.1). In network reliability theory, contraction is used to find upper bounds on the reliability function (which would correspond to upper bounds on the percolation probability function).

We begin with a few previously known relationships to illustrate the technique, then follow with derivations of new critical probability bounds for the pentagon, Kagomé and dice lattices in the following examples.

(*a*) Represent the square lattice S in the plane with vertex set Z^2 and edges between vertices that are a distance one apart. Contract all vertical edges with lower endpoints in the set $\{(x, y): y = x + 3k, k \in Z\}$. This produces the bowtie lattice B, for which the critical probability has been determined by Wierman (1984), so we have (see figure 1)

$$p_c(\mathrm{B}) \leq p_c(\mathrm{S}).$$

(*b*) Represent the hexagonal lattice H in the plane with regular hexagonal faces with one set of edges parallel to the x axis (called horizontal edges). The set of horizontal edges is partitioned into vertical columns of horizontal edges. By contracting the horizontal edges in every second column, we obtain a planar lattice which has



(*a*) (*b*)

Figure 1. (*a*) The square lattice. Edges marked with full circles are to be contracted. (*b*) A graph isomorphic to the bowtie lattice, obtained by contracting the edges marked in (*a*).

square and hexagonal faces, which is the dual of the 'bowtie lattice', denoted B^D. Therefore

$$p_c(B^D) \leq p_c(H).$$

Contracting each horizontal edge in B^D produces the square lattice S, so (see figure 2)

$$p_c(S) \leq p_c(B^D).$$

Let the square lattice S be represented in the plane as in example (*a*). Contract all vertical edges for which the lower endpoint has an even sum of coordinates. The resulting lattice is the triangular lattice T. Therefore

$$p_c(T) \leq p_c(S).$$

From examples (*a*) and (*b*), the contraction principle alone establishes the relationships

$$p_c(T) \leq p_c(S) \leq p_c(B^D) \leq p_c(H)$$

$$p_c(B) \leq p_c(S).$$

The inclusion principle alone establishes

$$p_c(T) \leq p_c(B) \leq p_c(S) \leq p_c(H)$$

$$p_c(S) \leq p_c(B^D).$$

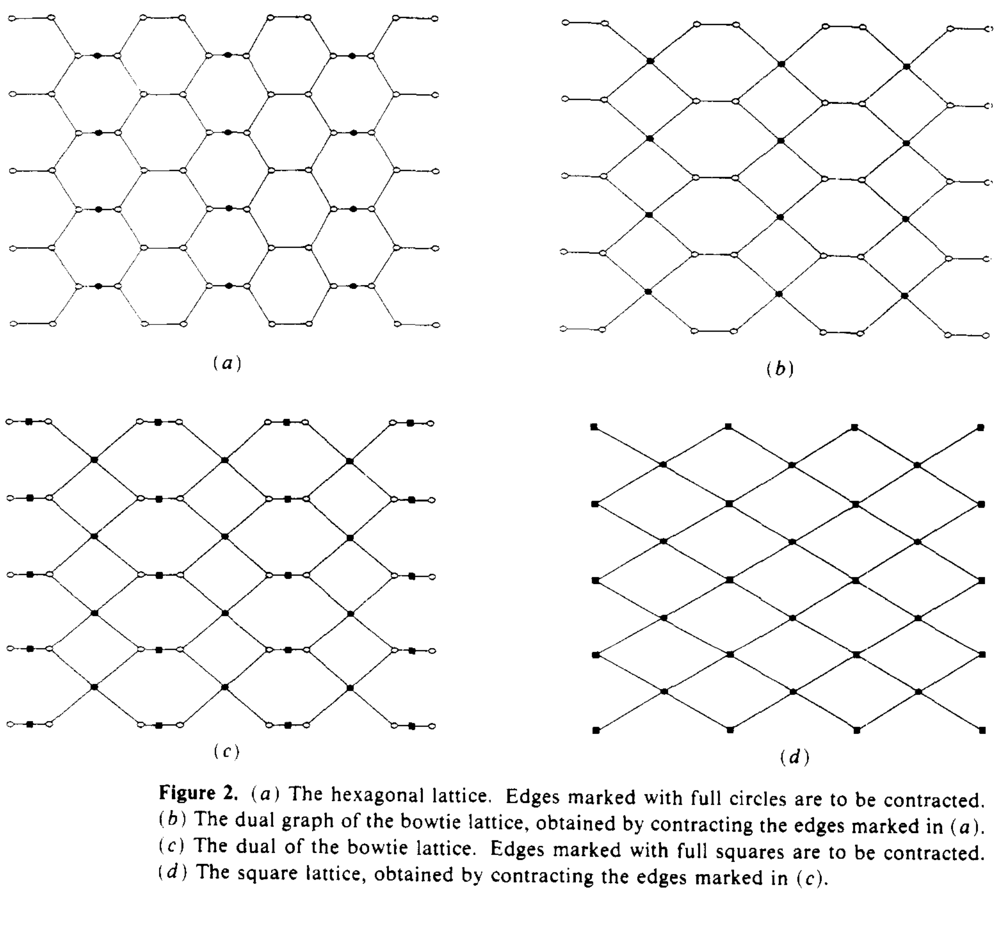


Figure 2. (*a*) The hexagonal lattice. Edges marked with full circles are to be contracted. (*b*) The dual graph of the bowtie lattice, obtained by contracting the edges marked in (*a*). (*c*) The dual of the bowtie lattice. Edges marked with full squares are to be contracted. (*d*) The square lattice, obtained by contracting the edges marked in (*c*).

However, B cannot be contracted to obtain T (since contracting an edge produces a vertex of degree 8 or 10) and H is not included in B^D (since deletion of any edge of B^D creates a face with more than six edges).

(*c*) Consider the representation of the hexagonal lattice in example (*a*). Contract all the edges with positive slope in every second column of such edges. Since we obtain the pentagon lattice P (see figure 3)

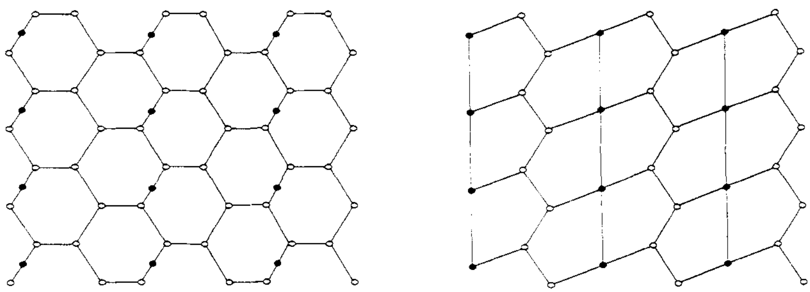
$$p_c(P) \leqslant p_c(H) \approx 0.6527.$$

(*d*) Represent the pentagon lattice as in figure 4. Contracting all edges with negative slope produces the square lattice, establishing

$$0.5000 = p_c(S) \leqslant p_c(P).$$

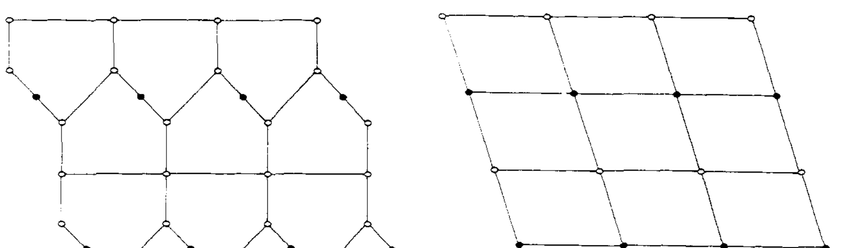
Waldor *et al* (1984) obtained the estimate $p_c(P) = 0.574$ using the annealed model solution for the zero-temperature bond-diluted Ising ferromagnet. Schulte and Sprenger (1985) obtained the estimate 0.579 ± 0.001 by computer simulation using the Hoshen–Kopelman algorithm. Note that both estimates are consistent with our bounds.

The results in examples (*c*) and (*d*) cannot be obtained by the inclusion principle. Removing an edge from S creates a face with six edges, so S does not contain P. Removing any edge from P creates a face with eight edges, so P does not contain H.



(*a*) (*b*)

Figure 3. (*a*) The hexagonal lattice. Edges marked with full circles are to be contracted. (*b*) The pentagon lattice, obtained by contracting the edges marked in (*a*).



(*a*) (*b*)

Figure 4. (*a*) The pentagon lattice. Edges marked with full circles are to be contracted. (*b*) The square lattice, obtained by contracting the edges marked in (*a*).

(*e*) Represent a lattice L in the plane with vertex set Z^2, with an edge between (i, j) and $(i, j+1)$ for all i and j, and an edge between (i, j) and $(i+1, j)$ for all i and all even j. This lattice may be viewed as a square lattice with a vertex inserted on each vertical edge. Thus, using the critical surface determined by Kesten (1982, §3.4, example (ii)), the bond percolation critical probability of L is the root of $p+p^2=1$, which is $(\sqrt{5}-1)/2 \approx 0.6180$.

Construct a lattice L′ from L by inserting an edge from $(i, 2j+1)$ to $(i+1, 2j+1)$ if $i+j$ is even. By the inclusion principle, we have that

$$p_c(\mathrm{L}') \leq p_c(\mathrm{L}).$$

Contracting the edges just inserted produces the Kagomé lattice K (see figure 5), so we obtain

$$p_c(\mathrm{K}) \leq p_c(\mathrm{L}') \leq 0.6180.$$

Since the dice and Kagomé lattices are dual graphs, the upper bound in example (*e*) provides the lower bound

$$0.3820 \leq p_c(\mathrm{D}).$$

There is disagreement in the literature concerning the value of the Kagomé lattice bond percolation critical probability. Early estimates of 0.449 ± 0.032 (Dean 1963) and 0.526 (Neal 1972) were obtained by Monte Carlo methods. A renormalisation group approach by Murase and Yuge (1979) yielded the estimate 0.4697. Ottavi (1979) used the star-triangle transformation to derive the bounds $0.522\,372 \leq p_c(\mathrm{K}) \leq 0.528\,924$. The argument has intuitive appeal but is not mathematically rigorous.

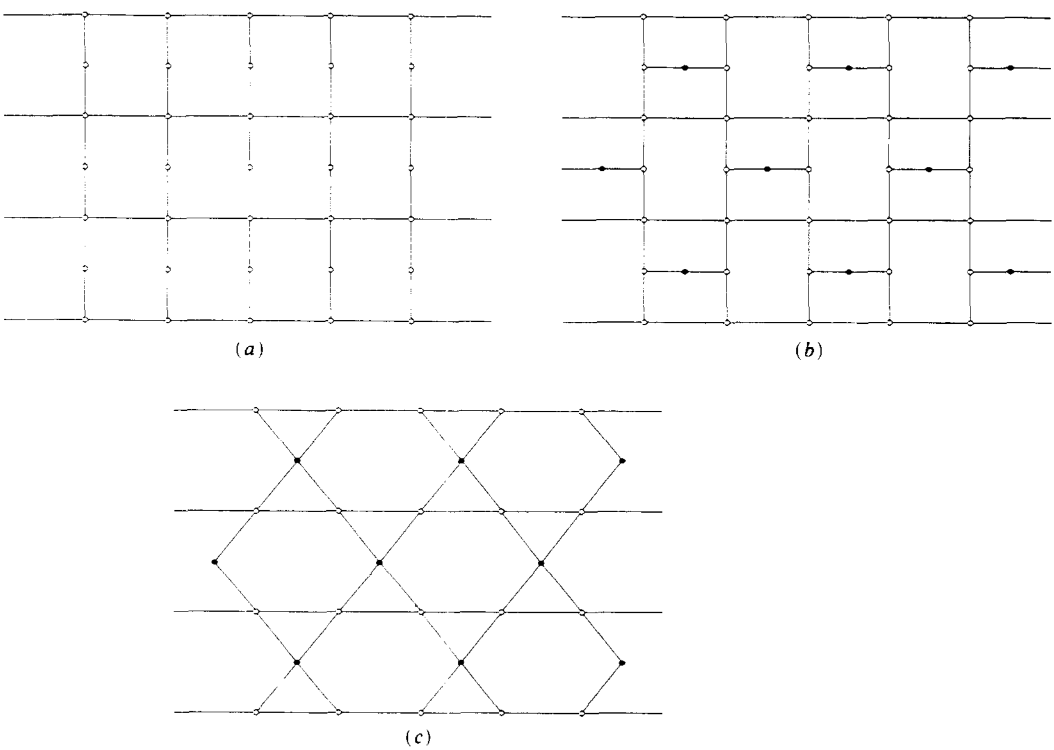


Figure 5. (*a*) The lattice L. (*b*) The lattice L′. Edges marked with full circles are to be contracted. (*c*) The Kagomé lattice, obtained by contracting the edges marked in (*b*).

Note that the upper bound given by the contraction principle is lower than the site percolation critical probability of the Kagomé lattice (≈ 0.6527, which is equal to the bond percolation critical probability of the hexagonal lattice), the best previous upper bound. It is unusual to find a rigorous bound between the site and bond model values.

If G^c is obtained from a planar graph G by contracting a set E of edges, then the dual graph $(G^c)^D$ is obtained from G^D by deleting the edges of G which cross edges in E. Thus, while several relationships shown above cannot be obtained from the inclusion principle alone, they may be obtained from the inclusion principle and Kesten's (1982) principal theorem for dual percolation models. However, the contraction principle provides a much more elementary proof and relationships may be recognised more easily for some graphs without working with the dual graphs.

We may consider a similar technique for site percolation models. Increasing the probability that a vertex v is open from p to 1 is equivalent to inserting edges between every pair of sites that are adjacent to v, i.e. to close packing a face which has its vertices at the sites that are adjacent to v. Examples of relationships between site percolation critical probabilities that can be obtained are: $p_c(\mathrm{T}) \leqslant p_c(\mathrm{H})$, $p_c(\mathrm{S}^\mathrm{M}) \leqslant p_c(\mathrm{B}) \leqslant p_c(\mathrm{B}^\mathrm{D})$ and $p_c(\mathrm{S}^\mathrm{M}) \leqslant p_c(\mathrm{S})$, where S^M denotes the matching graph of the square lattice. However, as yet, no improved numerical bounds have been obtained using the technique.

As in the case of the pentagon lattice, critical probability bounds rarely determine the leading digit of the critical probability. Although there have been many advances in percolation theory since its origins, determination of accurate bounds for critical probabilities for a variety of graphs remains an interesting and challenging open problem.

References

Broadbent S R 1954 *J. R. Stat. Soc.* B **16** 68
Broadbent S R and Hammersley J M 1957 *Proc. Camb. Phil. Soc.* **53** 629-41
Dean P 1963 *Proc. Camb. Phil. Soc.* **59** 397-410
Fisher M E 1961 *J. Math. Phys.* **2** 620-7
Hammersley J M 1957 *Ann. Math. Stat.* **28** 790-5
Kesten H 1980 *Commun. Math. Phys.* **74** 41-59
—— 1982 *Percolation Theory for Mathematicians* (Basel: Birkhauser)
Murase C and Yuge Y 1979 *J. Phys. A: Math. Gen.* **12** 1541-6
Neal D G 1972 *Proc. Camb. Phil. Soc.* **71** 97-106
Ottavi H 1979 *J. Physique* **40** 233-7
Schulte M and Sprenger P 1985 *J. Undergrad. Res. Phys.* **4** 15-6
Sykes M F and Essam J W 1964 *J. Math. Phys.* **5** 1117-27
Waldor M H, Wolff W F and Zittartz J 1984 *Phys. Lett.* **104A** 355-6
Wierman J C 1981 *Adv. Appl. Prob.* **13** 298-313
—— 1984 *J. Phys. A: Math. Gen.* **17** 1525-30